EXPANSIONS OF THE REALS WHICH DO NOT DEFINE THE NATURAL NUMBERS
VERSION 1.9

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ABSTRACT. We study first-order expansions of the reals which do not define the set of natural numbers. We also show that several stronger notions of tameness are equivalent to each others.

1. INTRODUCTION

O-minimal structures were introduced in the '80s as a framework for "tame topology" (see [vdD98b, vdDM96]). Many expansions of $\bar{\mathbb{R}}$ have been shown to be o-minimal: some important examples are $\langle \mathbb{R}_{an}, \exp \rangle$, the expansion of $\bar{\mathbb{R}}$ by restricted analytic functions and the exponential function (see [vdDMM94]), and the Pfaffian closure of $\langle \mathbb{R}_{an}, \exp \rangle$ (see [Spe99]).

Given a first-order structure $\mathbb{K}$, unless otherwise specified, by "definable" we will always mean "definable with parameters from $\mathbb{K}$". $\mathcal{R}$ will be an expansion of the field of real numbers $\bar{\mathbb{R}}$.

If $\mathcal{R}$ is o-minimal, then every definable set has finitely many connected components; most importantly, the dimension (see Definition 1.2) is well-behaved (that is, it satisfies the axioms (Dim 1–4) in [vdD89]: see also Conjecture 7.1), and moreover satisfies, for every nonempty definable set X,

(Dim 5) $\dim(X) = 0$ iff X is finite;
(Dim 6) $\dim(\partial X) < \dim(X)$.

In his article [Mil05], C. Miller studied several classes of structures expanding $\bar{\mathbb{R}}$ which still present a "tame" behaviour, without being o-minimal. In this article, we will focus on two such classes: **i-minimal** structures and **restrained** structures.

One of the first examples of restrained non o-minimal structures was given by d-minimal expansions of $\bar{\mathbb{R}}$ (see [vdD85, FM05, MT06]), that is structures such that every definable subset X of $\mathbb{R}$ with empty interior is the union of finitely many discrete sets, and the number of discrete sets does not depend on the parameters of definition of X: for instance, $\langle \bar{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$ is a d-minimal structure which is not o-minimal. D-minimal structures satisfy many of the properties of o-minimal structures; most importantly, the dimension is well-behaved (see [Mil05, For10a] and Theorem 1.13), but the additional properties (Dim 5) and (Dim 6) do not hold.

Remember that a subset X of a topological space Y is nowhere dense (in Y) if the closure of X has empty interior, and that a subset of $\mathbb{R}^n$ is null if it has Lebesgue measure 0.

Theorem 1.1. *T.f.a.e.:*

   (I) $\mathcal{R}$ *is **i-minimal** (that is, every definable subset of $\mathbb{R}$ has interior or is nowhere dense);*

   (II) *every definable subset of $\mathbb{R}$ has interior or is null;*

(III) *every definable subset of $\mathbb{R}$ has interior or has Hausdorff dimension 0.*

The above theorem shows that several "natural" strengthenings of d-minimality are equivalent to each other, and solves an open problem in [Mil05, §3.1] (and, implicitly, in [FKMS10] and [FM01], which also give some examples and general methods of constructing i-minimal expansions of $\bar{\mathbb{R}}$). See also [For10a, Mil05] for some more properties of i-minimal structures. We can give a higher-dimensional analogue of Theorem 1.1, but first we need to define the dimension.

Definition 1.2. Let $\mathbb{K}$ be an expansion of an ordered field, and $X \subseteq \mathbb{K}^n$. The **dimension** of X is $\dim(X)$, the maximum e such that there exists a coordinate space L of linear dimension e, such that $\Pi_L^n(X)$ has nonempty interior (inside L), where Π_L^n is the orthogonal projection onto L.

We denote by $\dim_{\mathcal{H}}$ the Hausdorff dimension.

Theorem 1.3. *Let $\mathcal{R}$ be i-minimal and $C \subseteq \mathbb{R}^n$ be definable and nonempty. Then, C is Lebesgue measurable, almost open, and $\dim_{\mathcal{H}}(C) = \dim(C)$.*

I am grateful to C. Miller for allowing me to present the following result and its proof (see §5 for the proof).

Theorem 1.4 (C. Miller). *Let $\mathcal{R}$ be an o-minimal expansion of $\bar{\mathbb{R}}$. Let $E \subset \mathbb{R}^n$ be closed, such that $\dim(E) = 0$. Denote by $\langle \mathcal{R}, E \rangle$ the expansion $\langle \mathcal{R}, (Y) \rangle$ of $\mathcal{R}$, where Y ranges among all subsets of finite Cartesian powers of E. Then, either $\langle \mathcal{R}, E \rangle$ defines $\mathbb{N}$, or $\langle \mathcal{R}, E \rangle^{\#}$ is i-minimal.*

An alternative way of extending o-minimality is given by structures with o-minimal open core.

Definition 1.5. The **open core** of $\mathbb{K}$ is the reduct of $\mathbb{K}$ generated by all open definable subset of $\mathbb{K}^n$ (for all $n \in \mathbb{N}$).

The main examples of structures with o-minimal open core are given by dense elementary pairs of o-minimal structures (see [vdD98a], but see also [DMS10, HG11] for other examples); their properties have been studied in [DMS10, For10b]. The dimension will no longer be "well-behaved" on such structures (for instance, the union of two definable sets of dimension 0 can have dimension 1), but its restriction to the class of sets definable in the open core will be well-behaved (but see again Conjecture 7.1).

All the above examples are particular cases of restrained structures.

Definition 1.6. We call $\mathcal{R}$ **restrained** if, for every definable discrete set $D \subset \mathbb{R}^n$ and every definable function $g : D \to \mathbb{R}$, $g(D)$ is nowhere dense (in $\mathbb{R}$).

See [BZ08, Hie11b] for more examples of restrained expansions of $\bar{\mathbb{R}}$.

On the one hand, non-restrained expansions of $\bar{\mathbb{R}}$ are "wild" (from a model-theoretic point of view; cf. §8(I)). If $\mathcal{R}$ defines the set $\mathbb{N}$, then $\mathcal{R}$ is not restrained (since, if we take $D := \mathbb{N} \times \mathbb{N}_{>0}$ and $f(x, y) := x/y$, we have $f(D)$ dense in $\mathbb{R}_{\geq 0}$). The converse is the main reason of interest in restrained structures:

Fact 1.7 ([Hie10, Theorem 1.1]). *Either $\mathcal{R}$ is restrained, or $\mathcal{R}$ defines $\mathbb{N}$.*

Corollary 1.8. *If the theory of $\mathcal{R}$ is decidable, then $\mathcal{R}$ is restrained.*

Notice that the converse of the above corollary is false: there exist o-minimal expansions of $\bar{\mathbb{R}}$ which are not decidable (for instance, it suffices to add to $\bar{\mathbb{R}}$ a constant for a suitable real number).

On the other hand, what we call restrained structures do present a tame behaviour: more precisely, a certain class of sets definable in restrained structures behaves in a controlled way.

Definition 1.9. Let $X \subseteq \mathbb{R}^n$ be definable (in $\mathcal{R}$). We say that X is a $\mathcal{D}_\Sigma$ set if there exists a definable closed set $Y \subseteq \mathbb{R}^{n+1}$, such that $\Pi_n^{n+1}(Y) = X$, where Π_n^{n+1} is the projection onto the first n coordinates.

Notice that if X is a Boolean combination of definable closed sets, then X is $\mathcal{D}_\Sigma$. Moreover, if $\mathcal{R}$ is d-minimal, then every definable set is a $\mathcal{D}_\Sigma$ set.

Theorem 1.10. *Let $\mathcal{R}$ be restrained. Let $D \subseteq \mathbb{R}^n$ be $\mathcal{D}_\Sigma$ and nonempty. T.f.a.e.:*
  (I) *D has empty interior;*
 (II) *D is nowhere dense;*
(III) *D is null.*
Moreover, $\dim_{\mathcal{H}}(D) = \dim(D)$.

While our main focus for this article is in expansions of $\bar{\mathbb{R}}$, the proofs are easier if we work in sufficiently saturated structures (see §4); moreover, many of our results extend to definably complete structures.

Definition 1.11. Let $\mathbb{K}$ be an expansion of an ordered field. $\mathbb{K}$ is **definably complete** (DC) if every definable subset of $\mathbb{K}$ has a least upper bound in $\mathbb{K} \cup \{\pm\infty\}$.

Definably complete structures were introduced in [Mil01], and have been studied (among other places) in [DMS10, Fra08, AF11, FS10, For10b, For11, For10a]; see also §2 for some properties of DC structures.

$\mathbb{K}$ will always be a DC structure (expanding an ordered field). We can generalize Definition 1.6 to arbitrary DC structures.

Definition 1.12. $\mathbb{K}$ is **restrained** if (it is a DC expansion of an ordered field and), for every definable discrete set $D \subset \mathbb{K}^n$ and every definable function $g : D \to \mathbb{K}$, $g(D)$ is nowhere dense (in $\mathbb{K}$).

The main result that allows us to prove the above theorems is the fact that the dimension function is well-behaved on $\mathcal{D}_\Sigma$ set (see Definition 2.4), provided that $\mathbb{K}$ is restrained.

Theorem 1.13. *Let $\mathbb{K}$ be restrained. Let $A, A' \subseteq \mathbb{K}^n$ be a $\mathcal{D}_\Sigma$ set.*
  (I) *$\dim(\overline{A}) = \dim(A)$ (where $\overline{A}$ is the topological closure of A).*
 (II) *Let $f : A \to \mathbb{K}^n$ be definable and continuous. Then, $\dim(f(A)) \leq \dim(A)$.*
(III) *$\dim(A \cup A') = \max\big(\dim(A), \dim(A')\big)$.*
(IV) *Let $B \subseteq \mathbb{K}^{n+m}$ be a $\mathcal{D}_\Sigma$ set. Let $A := \Pi_n^{n+m}(B)$. Assume that $\dim(A) = p$ and $\dim(B_{\bar{a}}) = q$ for every $\bar{a} \in A$ (where $B_{\bar{a}}$ is the fiber of B over $\bar{a}$). Then, $\dim(B) = p + q$.*

The proof of the above theorem is easier if we work inside ω-saturated structures (hence we formulated it for $\mathbb{K}$ instead of $\mathcal{R}$).

Some additional results on restrained structure, while not used in the proof of the previous theorems, may nevertheless be interesting on their own. In particular, we have the following further example of tame behaviour:

Theorem 1.14. *Let $\mathbb{K}$ be restrained. Let $f : \mathbb{K}^n \to \mathbb{K}^m$ be definable.*
  (I) *If f is continuous and $p \in \mathbb{N}$, then there exists a definable nowhere dense set $D \subset \mathbb{K}^n$, such that f is $\mathcal{C}^p$ outside D.*
 (II) *If f is $\mathcal{C}^1$, then Σ_f, the set of singular values of f, is nowhere dense.*

While for expansions of $\bar{\mathbb{R}}$ we have Fact 1.7, for general DC structures we have only a conjecture.

Conjecture 1.15. *Either $\mathbb{K}$ is restrained, or it defines a discrete subring (containing 1).*

Acknowledgements. Thanks to Philipp Lücke, Ben Miller, Chris Miller, and Tamara Servi.

2. Preliminaries

Proviso. $\mathbb{K}$ *will always be a DC structure expanding a field.*

We will freely use the results on definably complete structures contained in [Mil01] and [For10b, §2–5].

Notation 2.1. We denote by $\Pi_m^n : \mathbb{K}^n \to \mathbb{K}^m$ the projection onto the first m coordinates. Given a linear space $L \subseteq \mathbb{K}^n$, we denote by $\Pi_L^n : \mathbb{K}^n \to L$ the orthogonal projection onto L. Given $B \subseteq \mathbb{K}^{n+m}$ and $\bar{a} \in \mathbb{K}^n$, we denote by $B_{\bar{a}} \coloneqq \{\, \bar{c} \in \mathbb{K}^m : \langle \bar{a}, \bar{c} \rangle \in B \,\}$ the corresponding fiber of B.

In the proofs, we will always assume that there exists a pseudo-$\mathbb{N}$ set, that is a definable, discrete, closed, and unbounded subset of $\mathbb{K}$, which we will denote by N. If such a set does not exists, then $\mathbb{K}$ has locally o-minimal open core [For10b] (see also [DMS10]), and one can easily verify each result in that case.

Definition 2.2 ([FS10]). Let $X \subseteq \mathbb{K}^n$ be definable. We say that X is **definably meager** (in $\mathbb{K}^n$) if there is no family $\Xi \coloneqq \big(X_t : t \in \mathbb{K}\big)$ of closed subsets of $\mathbb{K}^n$, such that Ξ is definable, increasing, each X_t is nowhere dense, and $X \subseteq \bigcup_{t \in \mathbb{K}} X_t$.

In DC structures we have an analogue of Baire Category Theorem.

Fact 2.3. *For every $n \in \mathbb{N}$, $\mathbb{K}^n$ is not definably meager (in itself).*

Proof. See [Hie11a] and [FS10, Proposition 2.14]. Notice that for this article we are interested only in the case when $\mathbb{K}$ is restrained, where the proof of the above fact is much easier (see [For11, §6]). $\square$

We have a generalization of $\mathcal{D}_\Sigma$ sets to definably complete structures.

Definition 2.4. Let $X \subseteq \mathbb{K}^n$ be definable. We say that X is a $\mathcal{D}_\Sigma$ set if there exists a definable closed set $Y \subseteq \mathbb{K}^{n+1}$, such that $\Pi_n^{n+1}(Y) = X$.[1]

Fact 2.5 ([FS10]). *Let $X \subseteq \mathbb{K}^n$ be definable. Then, X is definably meager iff there exists $Y \subset \mathbb{K}^n$ such that Y is $\mathcal{D}_\Sigma$, with empty interior, and $X \subseteq Y$.*

We can already prove Theorem 1.13(III), which holds for arbitrary DC structures.

Lemma 2.6. *Let $A, A' \subseteq \mathbb{K}^n$ be $\mathcal{D}_\Sigma$. Then, $\dim(A \cup A') = \max\big(\dim(A), \dim(A')\big)$.*

Proof. Let $B \coloneqq A \cup A'$. It is clear that $\dim(B) \geq \max\big(\dim(A), \dim(A')\big)$. Assume, for a contradiction, that $\dim(B) > \max\big(\dim(A), \dim(A')\big)$. W.l.o.g., $\Pi_d^n(B)$ has nonempty interior. But $\Pi_d^n(B) = \Pi_d^n(A) \cup \Pi_d^n(B)$; by assumption and Fact 2.5, $\Pi_d^n(A)$ and $\Pi_d^n(A')$ are both definably meager; thus, $\Pi_d^n(B)$ is also definably meager, absurd. $\square$

Fact 2.7 ([FS10, Lemma 3.10]). *Let $f : \mathbb{K}^n \to \mathbb{K}^m$ be a definable function, such that its graph $\Gamma(f)$ is a $\mathcal{D}_\Sigma$ set. Then, $\mathfrak{D}(f)$, the set of discontinuity points of f, is definably meager.*

Fact 2.8 ([For10a, §2]). *Let $f : \mathbb{K}^n \to \mathbb{K}$ be such that f is the point-wise limit of a definable family of continuous functions. Then, $\mathfrak{D}(f)$ is definably meager.*

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[1] In [FS10] we called $\mathcal{D}_\Sigma$ sets "definably $\mathcal{F}_\sigma$ sets". However, since the main focus here is on expansions of $\mathbb{R}$, the previous nomenclature might be confusing (since a set which is definable and $\mathcal{F}_\sigma$ is not necessarily $\mathcal{D}_\Sigma$); hence, we adopt instead the nomenclature from [MS99], which, while being less suggestive, is also less prone to misunderstanding.

Definition 2.9. Let $X \subseteq \mathbb{K}^n$. We say that X is d-compact if X is definable, closed, and bounded.

D-compact sets are the definable analogue of compact subsets of $\mathbb{R}^n$ (see [Mil01]). their main property is given by the following fact.

Fact 2.10 ([Mil01])**.** *Let X be d-compact and $f : X \to \mathbb{K}^m$ be definable and continuous. Then, $f(X)$ is also d-compact.*

We use the following notation: given $m \geq n$, $A \subseteq \mathbb{K}^m$, and a map $\Pi : \mathbb{K}^m \to \mathbb{K}^n$, define $^c\Pi^c(A) := \mathbb{K}^n \setminus \big(\Pi(\mathbb{K}^m \setminus A)\big)$. The following remark help us to easily show that certain definable sets are $\mathcal{D}_\Sigma$, by looking at the formulae defining them.

Remark 2.11. 1) Let $\phi(\bar{x})$ be a formula of the form

$$(Q_1 y_1) \ldots (Q_n y_m)\psi(x, y_1, \ldots, y_m),$$

where $\psi(\bar{x}, \bar{y})$ is some formula, and each Q_i is a quantifier, either $\exists$ or $\forall$. Let $A \subseteq \mathbb{K}^n$ and $B \subseteq \mathbb{K}^{n+m}$ be the sets defined by ϕ and ψ respectively. Then, $A = \Pi_1^*(\ldots \Pi_m^*(B) \ldots)$, where each Π_i^* is either the orthogonal projection $\Pi : \mathbb{K}^{n+i} \to \mathbb{K}^{n+i-1}$ onto the first $(n+i-1)$ coordinates, or $^c\Pi^c$, according to whether Q_i is the quantifier $\exists$, or the quantifier $\forall$.
2) Let $B \subseteq \mathbb{K}^{n+m}$ be definable:

 i) If B is d-compact, then $\Pi(B)$ is d-compact;
 ii) If B is closed, or more generally a $\mathcal{D}_\Sigma$ set, then $\Pi(B)$ is a $\mathcal{D}_\Sigma$ set;
 iii) if B is open, then $\Pi(B)$ is also open;
 iv) if B is closed, then $^c\Pi^c(B)$ is also closed;
 v) if B is open, or more generally a definably $\mathcal{G}_\delta$ set (i.e., the complement of a $\mathcal{D}_\Sigma$ set), then $^c\Pi^c(B)$ is a definably $\mathcal{G}_\delta$ set.

It is not clear if the union of a definable family of $\mathcal{D}_\Sigma$ (resp., definably meager) sets, indexed by N, is $\mathcal{D}_\Sigma$ (resp., definably meager). To prove it we need some additional "uniformity" assumptions, which will be always satisfied in all our applications.

Definition 2.12. Let $\Xi := \big(X_i : i \in I\big)$ be a definable family of subsets of $\mathbb{K}^n$. We say that Ξ is a **strongly uniform** family of $\mathcal{D}_\Sigma$ sets (resp., of definably meager sets) if there exists a definable family $\big(Y_i : i \in I\big)$ of closed subsets of $\mathbb{K}^{n+1}$, such that, for every $i \in I$, $X_i = \Pi_n^{n+1}(Y_i)$ (resp., $X_i \subseteq \Pi_n^{n+1}(Y_i)$, and $\Pi_n^{n+1}(Y_i)$ has empty interior).

Remark 2.13. Let $\Xi := \big(X_n : n \in N\big)$ be a definable family of subsets of $\mathbb{K}^n$. If Ξ is a strongly uniform family of $\mathcal{D}_\Sigma$ (resp., definably meager) sets, then $\bigcup_{n \in N} X_n$ is also a $\mathcal{D}_\Sigma$ (resp., definably meager) set.

Lemma 2.14. *Let $A \subseteq \mathbb{K}^n$ be a $\mathcal{D}_\Sigma$ set, and $d := \dim(A)$. Then, the set $B := \{\, \bar{x} \in \mathbb{K}^d : \dim(A_{\bar{x}}) > 0 \,\}$ is a definably meager $\mathcal{D}_\Sigma$ set.*

Sketch of proof. The fact that B is definably meager is a consequence of the (definable version of) Kuratowski-Ulam theorem: see [MS99, 1.5(4)] and [FS10, §4] for details. If A is closed, then, by Remark 2.11, B is a $\mathcal{D}_\Sigma$ set. If A is definably $\mathcal{D}_\Sigma$, then $A = \bigcup_{i \in N} A(i)$, for some $\big(A(i) : i \in N\big)$ definable family of d-compact sets. Then, $B = \bigcup_{i \in N} B(i)$, where $B(i) := \{\, \bar{x} \in \mathbb{K}^d : \dim(A(i)_{\bar{x}}) > 0 \,\}$. By the previous case, each $A(i)$ is a $\mathcal{D}_\Sigma$ set. Moreover, as it is easy to check, the family $\big(B(i) : i \in N\big)$ is a strongly uniform family of $\mathcal{D}_\Sigma$ sets; thus, B is $\mathcal{D}_\Sigma$. $\qquad\square$

Lemma 2.15. *Let $A \subseteq \mathbb{K}^{n+m}$ be a $\mathcal{D}_\Sigma$ set. Let $B^{\geq p} := \{\, \bar{x} \in \mathbb{K}^d : \dim(A_{\bar{x}}) \geq p \,\}$. Then $B^{\geq p}$ is a $\mathcal{D}_\Sigma$ set.*

Proof. Assume that A is d-compact. Then, $B^{\geq p}$ is the union of finitely many sets, which, after a permutation of coordinates, are of the form

$$B(i) := \{\, \bar{b} \in \mathbb{K}^n : \exists \bar{c} \in \mathbb{K}^p \, \exists r > 0 \, \forall \bar{y} \in \mathbb{K}^p \, \left(\bar{y} \notin B(\bar{c}; r) \lor \exists \bar{z} \in \mathbb{K}^{n-p} \langle \bar{b}, \bar{y}, \bar{z} \rangle \in A \right) \,\}.$$

By Remark 2.11, each $B(i)$ is a $\mathcal{D}_\Sigma$ set, and thus $B^{\geq p}$ is also a $\mathcal{D}_\Sigma$ set.

If instead A is a $\mathcal{D}_\Sigma$ set, proceed as in the proof of Lemma 2.14. $\qquad\square$

Lemma 2.16. *Let $A \subseteq \mathbb{K}^n \times \mathbb{K}^p$ be a d-compact set. Let $B := \Pi_n^{n+p}(A)$. Define $f : B \to \mathbb{K}^p$, $f(x) := \operatorname{lex min}(A_x)$. Then, $\Gamma(f)$ is a definably $\mathcal{G}_\delta$ set, and $\mathfrak{D}(f)$ is definably meager.*

Proof. Let $\Pi := \Pi_n^{n+p}$. For simplicity, we will treat only the case $p = 1$. Notice that

$$\Gamma(f) = A \cap {}^c\Pi^c\{\, \langle \bar{x}, y, y' \rangle : y > y' \to \langle \bar{x}, y' \rangle \notin A \,\}.$$

The fact that $\Gamma(f)$ is definably $\mathcal{G}_\delta$ follows from Remark 2.11. $\mathfrak{D}(f)$ is definably meager by [DMS10, Lemma 2.8(1)]. $\qquad\square$

Notice that there are some compact subsets of $\mathbb{R}^2$, such that, for the corresponding function f as in the above lemma, $\Gamma(f)$ is a $\mathcal{G}_\delta$, but not an $\mathcal{F}_\sigma$ set.

Definition 2.17. [For11, §4] A definable set X is **at most pseudo-enumerable** if there exists a definable discrete set D and a definable surjective map $g : D \to X$.

Hence, $\mathbb{K}$ is restrained iff every at most pseudo-enumerable subset of $\mathbb{K}$ is nowhere dense.

Fact 2.18 ([For11, §5]).   (I) *Let $A \subseteq \mathbb{K}^n$ be at most pseudo-enumerable. Then, there exist $M \subset \mathbb{K}$ definable, closed and discrete, and $g : M \to \mathbb{K}^n$ definable, such that $A = g(M)$.*

  (II) *Let A, A' be at most pseudo-enumerable subsets of $\mathbb{K}^n$. Then, $A \cup A'$ and $A \times A'$ are also at most pseudo-enumerable.*

 (III) *Let A be at most pseudo-enumerable and $B \subseteq A$ be definable. Then, B is at most pseudo-enumerable.*

Lemma 2.19. *Let $X \subseteq \mathbb{K}$ be nowhere dense. Then, there exists a set $Y \subset \mathbb{K}$ discrete, definable, and such that $\overline{X} = \overline{Y}$. Moreover, the choice of Y can be made in a strongly uniform way: that is, if $Z \subset \mathbb{K}^{n+1}$ is definable, and for every $t \in \mathbb{K}^n$, Z_t is nowhere dense, then there exists $W \subset \mathbb{K}^{n+1}$ definable, such that, for every $t \in \mathbb{K}^n$, W_t is discrete, and $Z_t \subseteq \operatorname{cl}(W_t)$.*

Proof. W.l.o.g., we can assume that X is closed. Take Y to be the set of endpoints of $\mathbb{K} \setminus X$ (see [For11, §2]). $\qquad\square$

2.1. **Proof of Theorem 1.1.** The proof of Theorem 1.1 is surprisingly straightforward. (III $\Rightarrow$ II) is clear and (II $\Rightarrow$ I) is easy (see [Mil05, 3.1]).

(I $\Rightarrow$ III): let $A \subset \mathbb{R}$ be definable and with empty interior; we have to show that A has Hausdorff dimension 0. Since we assumed that $\mathcal{R}$ is i-minimal, we can substitute A with its closure, and thus, w.l.o.g., A is closed. By Lemma 2.19, there is a countable definable set $Y \subseteq A$ such that $A = \overline{Y}$. Assume, for a contradiction, that $\dim_{\mathcal{H}}(A) > 0$. By [EM01, Lemma 1], there exists a linear function $T : \mathbb{R}^n \to \mathbb{R}$, such that $T(A^n)$ has interior (in $\mathbb{R}$). Since Y is dense in A and T is continuous, $Z := T(Y^n)$ is somewhere dense; since $\mathcal{R}$ is i-minimal, Z has nonempty interior, contradicting the fact that Z is countable. $\qquad\square$

3. Meager and nowhere dense sets in restrained structures

Lemma 3.1. $\mathbb{K}$ *is restrained iff, for every definably meager set $X \subset \mathbb{K}$, we have that X is nowhere dense.*

Proof. For the "only if" direction, let $X \subseteq \bigcup_{i \in N} Y_i$, with $(Y_i : i \in \mathbb{N})$ definable family of nowhere dense sets. By Lemma 2.19, there exists a definable family of discrete sets $(Z_i : i \in N)$, such that, for every $i \in N$, $Y_i \subseteq \operatorname{cl}(Z_i)$. Let $W \coloneqq \bigcup_{i \in N} Z_i$. By [For11, §5], W is at most pseudo-enumerable. Since $X \subseteq \overline{W}$, we have that X is nowhere dense.

For the "if" direction, let $X \subset \mathbb{K}$ be at most pseudo-enumerable. Then, X is definably meager; thus, by assumption, X is nowhere dense, proving that $\mathbb{K}$ is restrained. $\square$

Proviso. *From now on, we will assume that $\mathbb{K}$ is restrained (besides being a definably complete expansion of an ordered field).*

The following lemma includes a higher-dimensional analogue of Lemma 3.1.

Lemma 3.2. (I) *Let $X \subseteq \mathbb{K}^n$ be definably meager. Then, X is nowhere dense.*
(II) *Let $X \subseteq \mathbb{K}^n$ be a $\mathcal{D}_\Sigma$ set. Then, either X has interior or it is nowhere dense.*
(III) *Let $A \subseteq \mathbb{K}^{n+p}$ be a $\mathcal{D}_\Sigma$ set. The set*

$$\mathfrak{B}_n(A) \coloneqq \{\, x \in \mathbb{K}^n : \operatorname{cl}(A)_x \neq \operatorname{cl}(A_x) \,\}$$

is nowhere dense.

Sketch of proof. (I) is equivalent to (II), since, if X is a $\mathcal{D}_\Sigma$ set, then $X \setminus \mathring{X}$ is a definably meager set and a $\mathcal{D}_\Sigma$ set.

The proofs of (II) and (III) proceed together by induction on n: the case $(\mathrm{II})_1$ is Lemma 3.1, while the proofs of $(\mathrm{III})_1$ and the inductive step are, with minor modifications, the same as in [MS99, 1.6]. $\square$

Definition 3.3 ([FS10, §5]). Let $X \subseteq \mathbb{K}^n$ be a definable. We say that X is **definably almost open**, or a.o. for brevity, if there exist a definable open set U and a definably meager set F, such that $X = U \,\Delta\, F$.

Fact 3.4 ([FS10, §5]). *The family of a.o. subset of $\mathbb{K}^n$ is a Boolean algebra containing all $\mathcal{D}_\Sigma$ sets.*

Lemma 3.5. *Let $X \subseteq \mathbb{K}^n$ be almost open. Then, X is nowhere dense; iff X has empty interior.*

Proof. The "if" direction is clear.. Conversely, assume that X has empty interior. By definition, $X = U \,\Delta\, F$, for some open set U and some definably meager set F. By Lemma 3.2, F is nowhere dense. Thus, since X has empty interior, U must be empty, and $X = F$. $\square$

Lemma 3.6 ($\mathcal{D}_\Sigma$-Uniformization). *Let $A \subseteq \mathbb{K}^n \times \mathbb{K}^p$ be a $\mathcal{D}_\Sigma$ set. Let $B \coloneqq \Pi_n^{n+p}(A)$. Then, there exists a definable function $f : B \to \mathbb{K}^p$, such that $\mathfrak{D}(f)$ is nowhere dense and, for all $b \in B$, $\langle b, f(b) \rangle \in A$.*

Proof of Lemma 3.6. Since A is a $\mathcal{D}_\Sigma$ set, there exists a definable family $\big(A(i) : i \in N\big)$ of d-compact sets, such that $A = \bigcup_{i \in N} A(i)$. For every $i \in N$, define $B(i) \coloneqq \Pi_n^{n+p}(A(i))$, and $C(i) \coloneqq B(i) \setminus \bigcup_{j < i} B(j)$. Notice that $B = \bigcup_{i \in N} C(i)$. For every $i \in N$, define $f_i : C_i \to \mathbb{K}^p$, $f_i(c) \coloneqq \operatorname{lex min}(A(i)_c)$, and define $f : B \to \mathbb{K}^p$ as $\Gamma(f) \coloneqq \bigcup_{i \in N} f_i$. If B has empty interior, then it is meager, and therefore nowhere dense, and we are done. Otherwise, let $U \coloneqq \bigcup_{i \in N} \operatorname{int}(C_i)$. Notice that $B \setminus U$ is definable and hence nowhere dense; thus, it suffices to show that $f \restriction U$ is continuous outside a nowhere dense set. By Lemma 2.16, each set $\mathfrak{D}(f_i)$ is definably meager,

and thus nowhere dense. Therefore, $F := \bigcup_i \big(\mathfrak{D}(f_i)\big)$ is also a definably meager set, and thus nowhere dense; let $U' := U \setminus \overline{F}$.

We claim that $f \upharpoonright U'$ is continuous. It suffices to show that $f \upharpoonright \operatorname{int}(C_i) \setminus \overline{F}$ is continuous, for every $i \in N$. However, $f \upharpoonright \operatorname{int}(C_i) \setminus \overline{F} = f_i \upharpoonright \operatorname{int}(C_i) \setminus \overline{F}$, and the latter is continuous by definition of F. $\qquad\square$

Corollary 3.7 ($\mathcal{D}_\Sigma$-Choice)**.** *Let $C \subseteq \mathbb{K}^n$ be a $\mathcal{D}_\Sigma$ set, and $f : C \to \mathbb{K}^p$ be a definable continuous function. Then, there exists a definable function $g : f(C) \to C$, such that, for every $y \in f(C)$, $f(g(y)) = y$, and $\mathfrak{D}(g)$ is nowhere dense.*

Proof. Let $A := \{ \langle f(c), c \rangle : c \in C \}$. Notice that A is a $\mathcal{D}_\Sigma$ set; the conclusion follows by applying Lemma 3.6 to A. $\qquad\square$

In the above corollary, notice that, if $f(C)$ itself is nowhere dense, then it might happen that $\mathfrak{D}(g) = f(C)$.

4. Dimension and closure operator

We will prove some good properties for the dimension function on $\mathcal{D}_\Sigma$ sets; in particular, we will prove Theorem 1.13.

Lemma 4.1. *Let $A \subseteq \mathbb{K}^n$ be a $\mathcal{D}_\Sigma$ set. Then, $\dim(\overline{A}) = \dim(A)$.*

Proof. Let $d := \dim(A)$. If $d = n$, the result is clear. Thus, w.l.o.g. $d < n$. Assume, for a contradiction, that $\dim(\overline{A}) = e > d$. Let L be a coordinate space of dimension e, such that $\Pi_L^n(\overline{A})$ has nonempty interior inside L. Notice that $\Pi_L^n(\overline{A}) \subseteq \operatorname{cl}(\Pi_L^n(A))$, that $\Pi_L^n(A)$ is a $\mathcal{D}_\Sigma$ set, and that, by assumption, $\Pi_L^n(A)$ has empty interior. Thus, by Lemma 3.2, $\Pi_L^n(A)$ is nowhere dense, and therefore $\Pi_L^n(\overline{A})$ has empty interior, absurd. $\qquad\square$

Notice that we cannot conclude that, if A is a $\mathcal{D}_\Sigma$ set, then $\dim(\partial A) < \dim(A)$, since the latter inequality fails for d-minimal structures.

Lemma 4.2. *Let $A \subset \mathbb{K}^n$ be a $\mathcal{D}_\Sigma$ set and $f : A \to \mathbb{K}^m$ be a definable continuous function. Assume that $\dim(A) = 0$. Then, $\dim(f(A)) = 0$.*

Proof. Assume, for a contradiction, that $\dim(f(A)) > 0$. W.l.o.g., $m = 1$. Since $f(A)$ is a $\mathcal{D}_\Sigma$ set, this means that $f(A)$ is nonmeager, and thus it contains an open interval I. By applying Corollary 3.7, we conclude that there exists an open interval $I' \subseteq I$ and a continuous definable function $g : I' \to \mathbb{K}^n$, such that, for every $y \in I'$, $g(y) \in A$ and $f(g(y)) = y$. Since I' is definably connected and g is continuous, $g(I')$ is also definably connected. Since $\dim(A) = 0$, the function g must be constant, contradicting $f(g(y)) = y$. $\qquad\square$

We write that a set is $\emptyset$-definable if it is definable without parameters. We introduce a matroid, which is useful in treating the dimension for $\mathcal{D}_\Sigma$ sets.

Definition 4.3. Let $B \subset \mathbb{K}^n$ be any set (not definable, in general) and $a \in \mathbb{K}$. We say that $a \in \operatorname{Fcl}(B)$ (the "F" stands for "$\mathcal{F}_\sigma$") if there exists $\bar{b} \in B^n$ and $X \subset \mathbb{K}^{n+1}$, such that:

(a) X is a $\mathcal{D}_\Sigma$ set and X is $\emptyset$-definable;
(b) For every $\bar{y} \in \mathbb{K}^n$, X_y is nowhere dense;
(c) $a \in X_{\bar{b}}$.

Notice that, in the above definition, under Assumption (a), Assumption (b) is equivalent to:

(b') For every $\bar{y} \in \mathbb{K}^n$, X_y is definably meager;

and to:

(b") For every $\bar{y} \in \mathbb{K}^n$, X_y has empty interior.

Lemma 4.4. Fcl *is a finitary matroid: that is (for every B and C subsets of $\mathbb{K}$ and every $a, c \in \mathbb{K}$)*

(I) Fcl *is extensive: $B \subseteq \mathrm{Fcl}(B)$;*
(II) Fcl *is increasing: $B \subseteq C$ implies $\mathrm{Fcl}(B) \subseteq \mathrm{Fcl}(C)$;*
(III) Fcl *is idempotent: $\mathrm{Fcl}(\mathrm{Fcl}(B)) = \mathrm{Fcl}(B)$;*
(IV) Fcl *satisfies the Exchange Property: $a \in \mathrm{Fcl}(Bc) \setminus \mathrm{Fcl}(B) \Rightarrow c \in \mathrm{Fcl}(Ba)$;*
(V) Fcl *is finitary: if $a \in \mathrm{Fcl}(B)$, then there exists $B' \subseteq B$ finite, such that $a \in \mathrm{Fcl}(B')$.*

Moreover, we have:

(VI) *let $X \subset \mathbb{K}^n$ be a $\mathcal{D}_\Sigma$ set of dimension $d \leq n$, which is $\emptyset$-definable. Let $\bar{b} \in X$. Then, there exists $\bar{b}'$ a subtuple of $\bar{b}$ of length d, such that $\mathrm{Fcl}(\bar{b}') = \mathrm{Fcl}(\bar{b})$.*

Proof. (II) and (V) are clear. (I) is also clear: take $X := \{\, \langle x, y \rangle \in \mathbb{K}^2 : x = y \,\}$ in the definition of Fcl.

(III) Let $a \in \mathrm{Fcl}(\mathrm{Fcl}(B))$. Thus, there exist $\bar{c} \in \mathbb{K}^n$ and $\bar{b} \in B^m$, such that $a \in \mathrm{Fcl}(\bar{b}\bar{c})$ and, for every $i = 1, \ldots, n$, $c_i \in \mathrm{Fcl}(\bar{b})$. Therefore, by definition, there exist $X \subset \mathbb{K}^{m+n+1}$ and $Y(i) \subset \mathbb{K}^{m+1}$, $i = 1, \ldots, n$, such that

(a) X and $Y(i)$ are $\mathcal{D}_\Sigma$ and $\emptyset$-definable set;
(b) For every $\bar{y} \in \mathbb{K}^n$ and $\bar{z} \in \mathbb{K}^m$, $X_{\bar{y},\bar{z}}$ and $Y(i)_{\bar{z}}$ are nowhere dense;
(c) $\langle \bar{b}, \bar{c}, a \rangle \in X$ and $\langle \bar{b}, c_i \rangle \in Y(i)$.

Define

$$Z := X \cap \{\, \langle \bar{y}, \bar{z}, x \rangle \in \mathbb{K}^m \times \mathbb{K}^n \times \mathbb{K} : \bigwedge_{i=1}^{n} \langle \bar{y}, z_i \rangle \in Y(i) \,\}, \qquad W := \Pi(Z),$$

where $\Pi : \mathbb{K}^m \times \mathbb{K}^n \times \mathbb{K} \to \mathbb{K}^m \times \mathbb{K}$ is the projection omitting the "middle" n coordinates.

Notice that $\langle \bar{b}, \bar{c}, a \rangle \in Z$, and therefore $\langle \bar{b}, a \rangle \in W$. Moreover, it is clear that W is a $\mathcal{D}_\Sigma$ and $\emptyset$-definable set. Thus, it suffices to show that, for every $\bar{y} \in \mathbb{K}^m$, $W_{\bar{y}}$ has empty interior. Assume, for a contradiction, that $W_{\bar{d}}$ has nonempty interior, for some $\bar{d} \in \mathbb{K}^m$. By $\mathcal{D}_\Sigma$-Uniformization, there exists an open interval $I \subseteq W_{\bar{d}}$ and a continuous function $g : I \to \mathbb{K}^n$, such that, for every $t \in I$, $\langle g(t), t \rangle \in Z_{\bar{d}}$. Since

$$\mathbb{Z}_{\bar{d}} \subseteq Y(1)_{\bar{d}} \times \cdots \times Y(n)_{\bar{d}} \times \mathbb{K},$$

and each $Y(i)_{\bar{d}}$ is nowhere dense, the function g must be constant, say $g(t) = \bar{b}'$. Thus, $\{\bar{b}'\} \times \mathbb{K} \subseteq Z_{\bar{d}}$, contradicting the assumption that $Z \subseteq X$ and (b).

(VI) We define the full dimension of X as the lexicographically ordered pair $\langle d, k \rangle$, where $d := \dim(X)$ and k is the number of d-dimensional coordinate spaces L, such that $\Pi_L^n(X)$ has nonempty interior. The proof is by induction on the full dimension of X. If $d = n$, we take $\bar{b}' = \bar{b}$. Otherwise, let $Y^{>0} := \{\, \bar{y} \in \mathbb{K}^d : \dim(X_{\bar{y}}) > 0 \,\}$, and $X^{>0} := X \cap \mathrm{cl}\big(\Pi^{-1}(Y^{>0})\big)$, where $\Pi := \Pi_d^n$. By Lemma 2.14, $X^{>0}$ is a $\mathcal{D}_\Sigma$ set of full dimension less than the full dimension of X. Thus, if $\bar{b} \in X^{>0}$, then the conclusion follows by inductive hypothesis. Assume instead that $\bar{b} \in X^0 := X \setminus X^{>0}$; define $\bar{b}' := \langle b_1, \ldots, b_d \rangle$. Consider for simplicity the case when $n = d + 1$; notice that X^0 is a $\mathcal{D}_\Sigma$ set with fibers of dimension 0: by definition, $b_n \in \mathrm{Fcl}(\bar{b}')$. In the general case we proceed similarly, by using Lemma 4.2, and conclude that $\bar{b} \in \mathrm{Fcl}(\bar{b}')$. The conclusion then follows from (I), (II) and (III).

(IV) Let $\bar{b} \subset B$ be of minimal length, such that $a \in \mathrm{Fcl}(\bar{b})$; say, $\bar{b} \in B^n$. There exists $X \subseteq \mathbb{K}^{n+2}$, such that X is a $\emptyset$-definable $\mathcal{D}_\Sigma$ set, for every $\langle \bar{y}, z \rangle \in \mathbb{K}^{n+1}$, $X_{\bar{y},z}$ is nowhere dense, and $\langle \bar{b}, c, a \rangle \in X$.

For every $\bar{d} \in \mathbb{K}^n$, define

$$Y_{\bar{d}}^{>0} := \{\, x \in \mathbb{K} : \dim\big(\{z \in \mathbb{K} : \langle \bar{d}, z, x \rangle \in X\}\big) > 0 \,\},$$

and $Y^{>0} \subseteq \mathbb{K}^{n+1}$ be the set whose fibers are given by the $Y_{\bar{d}}^{>0}$. By Lemma 2.14, each $Y_{\bar{d}}^{>0}$ is nowhere dense; moreover, the whole $Y^{>0}$ is a $\mathcal{D}_\Sigma$ set.

Assume that $a \in \mathrm{cl}(Y_{\bar{b}}^{>0})$. Notice that $Y^{>0}$ is a $\mathcal{D}_\Sigma$ set with empty interior, and therefore it is nowhere dense. Let $W := \mathrm{cl}(Y^{>0})$; notice that $\langle \bar{b}, a \rangle \in W$. Since W is a nowhere dense subset of $\mathbb{K}^{n+1}$, $\dim(W) \leq n$; thus, by Lemma 2.14, the set $C := \{\, \bar{d} \in \mathbb{K}^n : \dim(W_{\bar{d}}) > 0 \,\}$ is nowhere dense; by minimality of $\bar{b}$ and (VI), $\bar{b} \notin \overline{C}$. Let $W' := W \setminus (\overline{C} \times \mathbb{K})$. Then, $\langle \bar{b}, a \rangle \in W'$, and $W'_{\bar{d}}$ is nowhere dense for every $\bar{d} \in \mathbb{K}^n$; thus, $a \in \mathrm{Fcl}(\bar{b})$, a contradiction.

Assume now that $a \notin \mathrm{cl}(Y_{\bar{b}}^{>0})$. By Lemma 3.2, the set $B := \{\, \bar{d} \in \mathbb{K}^n : \mathrm{cl}(Y_{\bar{d}}^{>0}) \neq \mathrm{cl}(Y^{>0})_{\bar{d}} \,\}$ is nowhere dense; thus, by (VI) and minimality of $\bar{b}$, we have $\bar{b} \notin \overline{B}$, and therefore $a \notin \mathrm{cl}(Y^{>0})_{\bar{b}}$. Let

$$Z := \{\, \langle \bar{y}, z, x \rangle \in \mathbb{K}^n \times \mathbb{K} \times \mathbb{K} : \langle \bar{y}, z, x \rangle \in X \ \& \ \langle \bar{y}, x \rangle \notin \mathrm{cl}(Y^{>0}) \,\}.$$

Notice that Z is a $\emptyset$-definable $\mathcal{D}_\Sigma$ set, that $\langle \bar{b}, c, a \rangle \in Z$, and that, for every $\bar{d} \in \mathbb{K}^n$ and $a' \in \mathbb{K}$, $\dim(\{\, z \in \mathbb{K} : \langle \bar{d}, z, a' \rangle \in Z \,\}) \leq 0$. Thus, $c \in \mathrm{Fcl}(\bar{b}a)$. $\qquad\square$

Since, by Lemma 4.4, Fcl is a matroid, it induces a **rank** function, which we denote by rk.

Definition 4.5. Let $\bar{b} \in \mathbb{K}^n$. Let $\mathbb{K}(\bar{b})$ be the expansion of $\mathbb{K}$ by constants denoting $\bar{b}$. Let $\mathrm{Fcl}_{\bar{b}}$ be the matroid Fcl defined in $\mathbb{K}(\bar{b})$ and $\mathrm{rk}_{\bar{b}}$ be the rank associated to $\mathrm{Fcl}_{\bar{b}}$.

Remark 4.6. We have $\mathrm{Fcl}(\bar{b}) \subseteq \mathrm{Fcl}_{\bar{b}}(\emptyset)$. It is not true in general that $\mathrm{Fcl}(\bar{b}) = \mathrm{Fcl}_{\bar{b}}(\emptyset)$. Therefore, $\mathrm{rk}_{\bar{b}}(\bar{a}) \leq \mathrm{rk}(\bar{a}/\bar{b})$. The relative (in $\mathbb{K}$) field algebraic closure of $\bar{b}$ is contained in $\mathrm{Fcl}(\bar{b})$, but the model-theoretic algebraic closure of $\bar{b}$ might not be contained in $\mathrm{Fcl}(\bar{b})$. On the other hand, $\mathrm{Fcl}_{\bar{b}}(\emptyset)$ contains the model-theoretic algebraic closure of $\bar{b}$.

Lemma 4.7. *Assume that $\mathbb{K}$ is ω-saturated.*

(I) *Let $U \subseteq \mathbb{K}^n$ be open, nonempty, and definable. Then, there exists $\bar{b} \in U$ such that $\mathrm{rk}(\bar{b}) = n$.*

(II) *Let $X \subseteq \mathbb{K}^n$ be a $\emptyset$-definable $\mathcal{D}_\Sigma$ set. Then, $\dim(X) = \max(\mathrm{rk}(\bar{b}) : \bar{b} \in X)$.*

Proof. (I) We proceed by induction on n. First, we consider the case $n = 1$. Consider the following partial type (over the parameters of definition of U):

$$\Lambda(x) := \{\, x \in U \ \& \ x \notin Y : Y \subset \mathbb{K} \text{ nowhere dense and } \emptyset\text{-definable} \,\}.$$

If $\Lambda(x)$ is consistent, then any realization b of $\Lambda(x)$ will satisfy $b \in U$ and $\mathrm{rk}(b) = 1$. If, for a contradiction, $\Lambda(x)$ were inconsistent, there would exists finitely many nowhere dense sets $X_1, \ldots, X_k \subset \mathbb{K}$, such that $U \subseteq X_1 \cup \cdots \cup X_k$, which is absurd.

Assume now that we have proved (I) for $n - 1$; we want to prove it for n. Let $V := \Pi_{n-1}^n(U)$. By inductive hypothesis, there exists $\bar{c} \in V$ such that $\mathrm{rk}(\bar{c}) = n-1$. Add $\bar{c}$ to the language, and consider the matroid $\mathrm{Fcl}_{\bar{c}}$. By applying the case $n = 1$ to the open set $U_{\bar{c}}$ and the matroid $\mathrm{Fcl}_{\bar{c}}$, we find $b_n \in U_{\bar{c}}$, such that $b_n \notin \mathrm{Fcl}_{\bar{c}}(\emptyset)$. Therefore, $b_n \notin \mathrm{Fcl}(\bar{c})$. Let $\bar{b} := \langle \bar{c}, b_n \rangle \in \mathbb{K}^n$. We have that $\bar{b} \in U$ and $\mathrm{rk}(\bar{b}) = n$.

(II) Let $d := \dim(X)$ and $e := \max(\mathrm{rk}(\bar{b}) : \bar{b} \in X)$. We prove that $d \geq e$ and $e \geq d$.

($d \geq e$). Let $\bar{b} \in X$ such that $\mathrm{rk}(\bar{b}) = e$. By Lemma 4.4(6), there exists $\bar{b}'$ a subtuple of $\bar{b}$ of length d, such that $\mathrm{Fcl}(\bar{b}') = \mathrm{Fcl}(\bar{b})$. Thus, $e = \mathrm{rk}(\bar{b}) = \mathrm{rk}(\bar{b}') \leq d$.

($e \geq d$). Since $\dim(X) = d$, w.l.o.g. $\Pi_d^n(X)$ contains a nonempty definable open set U. By (I), there exists $\bar{c} \in U$ such that $\mathrm{rk}(\bar{c}) = d$. Any $\bar{b} \in X \cap (\{\bar{c}\} \times \mathbb{K}^{n-d})$ will satisfy $\mathrm{rk}(\bar{b}) \geq d$. $\qquad\square$

Lemma 4.8. *Let $A \subseteq \mathbb{K}^{m+n}$ be $\mathcal{D}_\Sigma$ and $B := \Pi_n^{n+m}(A)$.*

(I) *Assume that* $\dim(B) \geq q$ *and, for every* $\bar{b} \in B$, $\dim(A_{\bar{b}}) \geq p$. *Then,* $\dim(A) \geq p + q$.

(II) *Assume that* $\dim(B) \leq q$ *and, for every* $\bar{b} \in B$, $\dim(A_{\bar{b}}) \leq p$. *Then,* $\dim(A) \leq p + q$.

(III) *In particular, if* $\dim(A_{\bar{b}}) = p$ *for every* $\bar{b} \in B$, *then* $\dim(A) = \dim(B) + p$.

Proof. W.l.o.g., $\mathbb{K}$ is ω-saturated and A is $\emptyset$-definable.

(I) By Lemma 4.7, there exists $\bar{b} \in B$ such that $\mathrm{rk}(\bar{b}) \geq q$. By Lemma 4.7 again, applied to the matroid $\mathrm{Fcl}_{\bar{b}}$, there exists $\bar{c} \in A_{\bar{b}}$, such that $\mathrm{rk}_{\bar{b}}(\bar{c}) \geq p$. Thus,

$$\mathrm{rk}(\bar{b}\bar{c}) = \mathrm{rk}(\bar{b}) + \mathrm{rk}(\bar{c}/\bar{b}) \geq \mathrm{rk}(\bar{b}) + \mathrm{rk}_{\bar{b}}(\bar{c}) \geq p + q.$$

The conclusion follows by applying Lemma 4.7 a third time.

(II) First, we do the case when $p = 0$. Let $\langle \bar{b}, \bar{c} \rangle \in A$; by Lemma 4.7, it suffices to show that $\mathrm{rk}(\bar{b}\bar{c}) \leq p$. Since $p = 0$, $\bar{c} \in \mathrm{Fcl}(\bar{b})$, and therefore $\mathrm{rk}(\bar{b}\bar{c}) = \mathrm{rk}(\bar{b})$. By Lemma 4.7, $\mathrm{rk}(\bar{b}) \leq p$, and we are done.

Next, we do the case when $p = m$. It suffices to show that, for every $\langle \bar{b}, \bar{c} \rangle \in B \times \mathbb{K}^p$, $\mathrm{rk}(\bar{b}\bar{c}) \leq p + q$. Since $\bar{b} \in B$, $\mathrm{rk}(\bar{b}) \leq q$. Since $\bar{c} \in \mathbb{K}^p$, $\mathrm{rk}(\bar{c}) \leq p$. Thus,

$$\mathrm{rk}(\bar{b}\bar{c}) = \mathrm{rk}(\bar{b}) + \mathrm{rk}(\bar{c}/\bar{b}) \leq \mathrm{rk}(\bar{b}) + \mathrm{rk}(\bar{c}) \leq p + q.$$

Now we do the general case by induction on q and p. Define $\Pi(m, p)$ be the set of orthogonal projection from $\mathbb{K}^m$ to some p-dimensional coordinate space. For every $\pi \in \Pi(m, p)$, let $B(\pi) := \{ \bar{b} \in B : \dim(\pi(A_{\bar{b}})) \geq p \}$, and $B(0) := B \setminus \bigcup_{\pi \in \Pi(m,p)} \mathrm{cl}(B(\pi))$. Correspondingly, $A(\pi) := A \cap (B(\pi) \times \mathbb{K}^m)$ and $A(0) := A \cap (B(0) \times \mathbb{K}^m)$. Let $\Pi'(m, p) := \Pi(n, m) \cup \{ 0 \}$. For every $\pi \in \Pi'(m, p)$, $A(\pi)$ is a $\mathcal{D}_\Sigma$ set. Moreover, $A \subseteq \bigcup_{\pi \in \Pi'(m,p)} \mathrm{cl}(A(\pi))$; thus, it suffices to prove that, for every $\pi \in \Pi'(m, p)$, $\dim(\mathrm{cl}(A(\pi))) \leq p + q$. Thus, by Lemma 4.1, it suffices to show that, for every $\pi \in \Pi'(m, p)$,

$$\dim(A(\pi)) \leq p + q. \tag{1}$$

Fix $\pi \in \Pi'(m, p)$. If $\dim(B(\pi)) < q$, then (1) follows by induction on q; therefore, we can assume $\dim(B(\pi)) = q$.

If $\pi = 0$, then, by definition of dimension, $\dim(A(0)_{\bar{b}}) \leq p - 1$, for every $\bar{b} \in B(0)$. Therefore, by induction on p we have $\dim(A(0)) \leq p + q - 1$.

Assume now that $\pi \in \Pi(m, p)$, and consider $A(\pi)$; w.l.o.g., $\pi = \Pi_p^m$. Notice that the assumption in (II) implies that, for every $\bar{b} \in B(\pi)$, $\dim(A(\pi)_{\bar{b}}) = p$. Let $D(\pi) := \Pi_{n+p}^{n+m}(A(\pi))$. Let $\bar{b} \in B(\pi)$. Notice that $D(\pi)_{\bar{b}} = \pi(A(\pi)_{\bar{b}})$. By the case $m = p$, $\dim(D(\pi)) \leq p + q$, and, by (I), we have $\dim(D(\pi)) = p + q$. Let

$$D(\pi)'_{\bar{b}} := \{ \bar{e} \in D(\pi)_{\bar{b}} : \dim(A(\pi)_{\bar{b},\bar{e}}) > 0 \}, \qquad A(\pi)' := A(\pi) \cap (D(\pi)' \times \mathbb{K}^{m-p}).$$

Let $D(\pi)'' := D(\pi) \setminus \mathrm{cl}(D(\pi)')$ and $A(\pi)'' := A(\pi) \cap (D(\pi)'' \times \mathbb{K}^{m-p})$. By Lemma 2.14, $A(\pi)'$ and $A(\pi)''$ are both $\mathcal{D}_\Sigma$ sets, and, by Lemma 4.1, it suffices to show that $\dim(A(\pi)') \leq p + q$ and $\dim(A(\pi)'') \leq p + q$.

For every $\bar{d} \in D(\pi)''$, $\dim(A_{\bar{d}}) = 0$; thus, by the case $p = 0$, applied to $A(\pi)''$ and $D(\pi)''$ instead of A and B, $\dim(A(\pi)'') = \dim(D(\pi)'') \leq \dim(D(\pi)) \leq p + q$.

Let $R(A) = R(A; n, m, p)$ be the set of projections $\rho \in \Pi(m, p)$, such that $\dim(\rho(A_{\bar{b}})) \geq p$ for at least one $\bar{b} \in B$. We conclude by induction on the cardinality of $R(A)$. By Lemma 2.14, $\dim(D(\pi)'_{\bar{b}}) < p$ for every $\bar{b} \in B$. Notice that $D(\pi)'_{\bar{b}} = \pi(A(\pi)'_{\bar{b}})$, and therefore $\pi \in R(A) \setminus R(A(\pi)')$; by our inductive hypothesis, $\dim(A(\pi)') \leq p + q$, and we are done. $\square$

Corollary 4.9. *Let* $B \subseteq \mathbb{K}^n$ *be a* $\mathcal{D}_\Sigma$ *set. Let* $f : B \to \mathbb{K}^m$ *be definable and continuous. Then,* $\dim(f(B)) \leq \dim(B)$. *If, moreover,* f *is finite-to-one, then* $\dim(f(B)) = \dim(B)$.

Proof. Apply Lemma 4.8 to $A := \Gamma(f)$. $\square$

5. The real case

In this section we will prove most of the theorems in the introduction which are specific to expansions of $\bar{\mathbb{R}}$ (since they mention Lebesgue measure and Hausdorff dimension). Given a set $X \subseteq \mathbb{K}^n$, we define $X - X := \{\, x - y \,:\, x, y \in X \,\}$. Remember that $\mathcal{R}$ is a restrained expansion of the real field.

Lemma 5.1. *Let $C \subseteq \mathbb{R}$ be a nonempty $\mathcal{D}_\Sigma$ set. Then, $\dim(C) = \dim_{\mathcal{H}}(C)$.*

Proof. For every set, $\dim_{\mathcal{H}}(C) \geq \dim(C)$. Thus, we have to prove that if C is nowhere dense, then $\dim_{\mathcal{H}}(C) = 0$. Since $\dim(C) = \dim(\overline{C})$, w.l.o.g. C is closed. Conclude as in the proof of Theorem 1.1. $\square$

Lemma 5.2. *Let $D \subseteq \mathbb{R}^n$ be a $\mathcal{D}_\Sigma$ set. Then, $\dim(D) < n$ iff $\overline{D}$ is null.*

Proof. The "if" direction is clear. So, assume that D is nowhere dense; we have to show that $\overline{D}$ is null. We proceed by induction on n. The case $n = 1$ is Lemma 5.1. Assume that we have already proven the conclusion for $n - 1$. Assume that D is nowhere dense. Since $\dim(\overline{D}) = \dim(D)$, w.l.o.g. D is closed. Let $F := \{\, \bar{c} \in \mathbb{K}^{n-1} : D_{\bar{c}}$ has positive Lebesgue measure $\}$. By the case $n = 1$, $F = \{\, \bar{c} \in \mathbb{K}^{n-1} : \dim(D_{\bar{c}}) = 1 \,\}$. By Lemma 2.14, since $\dim(D) \leq n - 1$, we have that F is nowhere dense in $\mathbb{K}^{n-1}$. By inductive hypothesis, F is null. By Fubini's theorem, D is null, and we are done. $\square$

We now prove the "moreover" clause in Theorem 1.10. We employ techniques similar to the one used in [EM01, Lemma 1]. We need some preliminary results from geometric measure theory.

Fact 5.3 ([Mat95, Theorem 8.10]). *Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be $\mathcal{F}_\sigma$ sets. Then, $\dim_{\mathcal{H}}(A \times B) \geq \dim_{\mathcal{H}}(A) + \dim_{\mathcal{H}}(B)$.*

Fact 5.4 ([Mat95, Corollary 9.8]). *Let $A \subseteq \mathbb{R}^n$ be an $\mathcal{F}_\sigma$ set. Assume that $\dim_{\mathcal{H}}(A) > m$ (with $m < n$). Then, there exists an m-dimensional linear subspace $L \subseteq \mathbb{R}^n$, such that $\Pi^n_L(A)$ is not null.*

Lemma 5.5. *Let $X \subseteq \mathbb{R}^n$ be a nonempty $\mathcal{D}_\Sigma$ set. Then, $\dim(X) = \dim_{\mathcal{H}}(X)$.*

Proof. **Claim** 1. Let $Y \subseteq \mathbb{R}^m$ be a $\mathcal{D}_\Sigma$ set, such that $\dim(Y) = p$. Then, $\dim_{\mathcal{H}}(Y) \leq p + 1$.

We proceed by induction on m and p. If $p \geq m - 1$, the result is clear. Thus, w.l.o.g. $m > p + 1 \geq 1$. Assume, for a contradiction, that $\dim_{\mathcal{H}}(Y) > p + 1$. Thus, by Fact 5.4, $Y' := \Pi^n_L(Y)$ is not null, for some $(p + 1)$-dimensional linear subspace $L \subseteq \mathbb{R}^m$. By Lemma 5.2, $\dim(Y') = p + 1$; since $\dim(Y) \geq \dim(Y')$, we have a contradiction, and the claim is proven.

The inequality $\dim_{\mathcal{H}}(X) \geq \dim(X)$ is true for any set $X \subseteq \mathbb{R}^n$; thus, it suffices to prove the opposite inequality. Assume, for a contradiction, that $\dim_{\mathcal{H}}(X) = \dim(X) + \varepsilon$, with $\varepsilon > 0$. Let $m \in \mathbb{N}$ such that $m > 1/\varepsilon$. Let $Y := X^m \subseteq \mathbb{R}^{nm}$. By Fact 5.3, $\dim_{\mathcal{H}}(Y) \geq n \dim_{\mathcal{H}}(X) \geq np + 1 = \dim(Y) + 1$, contradicting the claim. $\square$

Proof of Theorem 1.3. Let $D := \overline{C}$. By Lemma 5.5 (applied to D), $\dim_{\mathcal{H}}(D) = \dim(D) = \dim(C) \leq \dim_{\mathcal{H}}(C) \leq \dim_{\mathcal{H}}(D)$. We also have to show that C is definably almost open and Lebesgue measurable. Notice that $E := C \setminus \mathring{C}$ is definable and has empty interior. Thus, by [For10a, §3], E is nowhere dense. Moreover, by Lemma 5.2 E is null. Thus, $C = \mathring{C} \cup E$, where $\mathring{C}$ is a definable open set and E a definable set which is nowhere dense and null. $\square$

Proof of Theorem 1.4. Assume that $\langle \mathcal{R}, E \rangle$ does not define $\mathbb{N}$ (thus, $\langle \mathcal{R}, E \rangle$ is restrained). Let $B \subset \mathbb{R}$ be definable in $\langle \mathcal{R}, E \rangle^{\#}$ and assume that B has no interior. We must prove that B is nowhere dense.

First, we do the case when $n = 1$. By [FKMS10, 1.11], there exists $f : \mathbb{R}^m \to \mathbb{R}$ definable in $\mathcal{R}$, such that $B \subseteq \mathrm{cl}(f(E^m))$. Since we assumed that E is closed and 0-dimensional and $\langle \mathcal{R}, E \rangle$ is restrained, we have that $\dim(f(E^n)) = 0$; thus, $f(E^n)$ is nowhere dense, and we are done.

Assume now that $n \geq 1$. Let F be the union of the closures of the coordinate projections of E: notice that F is closed, nowhere dense, and definable in $\langle \mathcal{R}, E \rangle$, and that $\langle \mathcal{R}, E \rangle^{\#} = \langle \mathcal{R}, F \rangle^{\#}$. We conclude by applying the case $n = 1$ to F. $\square$

Question 5.6 (C. Miller)**.** Let $\mathcal{R}$ be an expansion of $\bar{\mathbb{R}}$. Are the following equivalent?

(I) $\mathcal{R}$ is i-minimal;
(II) every definable subset of $\mathbb{R}$ either has interior or has Minkowski upper dimension 0.

In the above question, it is clear that (II) implies (I); however, we don't even know if d-minimal structures satisfy (II).

6. Continuous and differentiable functions

Notice that we have little control on general definable functions: there are examples of restrained expansions of $\mathbb{R}$ which define a function $f : \mathbb{R} \to \mathbb{R}$ whose graph is dense in $\mathbb{R}^2$, and such that f is discontinuous at every point: see [vdD98a, p. 62]. However, continuous functions are much better behaved.

Lemma 6.1. *Let $C \subseteq \mathbb{K}^n$ be definable and nowhere dense. Let $f : \mathbb{K}^n \to \mathbb{K}^m$ be definable and continuous. If $m \geq n$, then $f(C)$ is nowhere dense.*

Proof. Let $B := \overline{C}$. By definition, $\dim(B) < m$. By Corollary 4.9, $\dim(f(B)) < m$, and therefore $f(B)$ is nowhere dense. $\square$

Thus, a Peano curve is not definable.

Corollary 6.2. *Let $f : \mathbb{K}^n \to \mathbb{K}^m$ be definable and continuous. If $m > n$, then $f(\mathbb{K}^n)$ is nowhere dense.*

6.1. **Differentiable functions.** Definable continuous functions are differentiable almost everywhere.

Lemma 6.3. *Let $f : \mathbb{K} \to \mathbb{K}$ be definable and continuous. Fix $p \in \mathbb{N}$. Then, there exists $V \subseteq \mathbb{K}$ which is open, definable, and dense, such that,*

(I) *for every I definably connected component of V, $f \upharpoonright I$ is either constant, or strictly monotone;*
(II) *f is $\mathcal{C}^p$ on V.*

Proof of Lemma 6.3. The proof of (I) follows easily as in [Mil01, Thm. 3.3], by using $\mathcal{D}_{\Sigma}$-Choice.

Now we prove (II). It suffices to prove the case when $p = 1$. Let F be the closure of

$$\{ \langle t, x, z \rangle : t \neq 0, z = \frac{f(x+t) - f(x)}{t} \}$$

inside $\mathbb{K} \times \mathbb{K} \times \mathbb{K}_\infty$, where $\mathbb{K}_\infty = \mathbb{K} \cup \{ \pm\infty \}$. Let $G := F_0 = \{ \langle x, z \rangle : \langle 0, x, z \rangle \in F \}$. Define the functions $g_l, g_r : \mathbb{K} \to \mathbb{K}_\infty$, $g_l(x) := \min(G_x)$, $g_r(x) := \max(G_x)$. By Lemma 2.16, g_r and g_r are continuous outside a nowhere dense definable set D. Let I be a definably connected component of $\mathbb{K} \setminus \overline{D}$. It suffices to show that, after maybe ignoring a nowhere dense definable set, $g_l = g_r$ on I and that they are finite.

Let $X := \{\, x \in I : g_r(x) = +\infty \,\}$. Notice that X must have empty interior, and thus it is nowhere dense. Similarly, the set $\{\, x \in I : g_r(x) = -\infty \,\}$ is nowhere dense. Thus, after shrinking I, we can assume that g_r and g_l assume only finite values on I. Let $Y := \{\, x \in I : g_l(x) < g_r(x) \,\}$. It suffices to show that Y has empty interior. Assume not: by continuity, and since $g_l \leq g_r$, there exists an open subinterval $J \subseteq I$ and a constant $c \in \mathbb{K}$, such that, for every $x \in J$, $g_l(x) < c < g_r(x)$. Consider now the function $h(x) := f(x) - cx$, $h : J \to \mathbb{K}$. By (I), after maybe shrinking J to a smaller subinterval, h is either constant or strictly monotone. However, h constant contradicts $g_r > c$, h strictly increasing contradicts $g_l < c$, and h strictly decreasing contradicts $g_r > c$. $\qquad\square$

Exercise 6.4. Let $f : \mathbb{K} \to \mathbb{K}$ be definable and monotone (but not necessarily continuous). Then, there exists a closed definable nowhere dense set F such that f is $\mathcal{C}^1$ outside F.

Proof of Theorem 1.14. (I) Same proof as in Lemma 6.3(II).

(II) Notice that Σ_f is a $\mathcal{D}_\Sigma$ set. The result follows easily from $\mathcal{D}_\Sigma$-choice: see [For10a, §4] for details. $\qquad\square$

7. FURTHER CONJECTURES

Conjecture 7.1. *Let $\mathbb{K}$ be restrained. There is a dimension function d in the sense of in the sense of [vdD89]. That is, d is a function assigning to every definable set a number in $\mathbb{N} \cup \{\, -\infty \,\}$, satisfying the following axioms: for every definable sets A and $B \subseteq \mathbb{K}^n$ and $C \subseteq \mathbb{K}^{n+1}$,*

(Dim 1) $d(A) = -\infty$ *iff* $A = \emptyset$, $d(\{\, a \,\}) = 0$ *for each* $a \in \mathbb{K}$, $d(\mathbb{K}) = 1$.
(Dim 2) $d(A \cup B) = \max(d(A), d(B))$;
(Dim 3) $d(A^\sigma) = d(A)$ *for each permutation* σ *of* $\{\, 1, \ldots, n \,\}$.
(Dim 4) *Define* $C(i) := \{\, \bar{x} \in \mathbb{K}^n : d(C_{\bar{x}}) = i \,\}$, $i = 0, 1$. *Then, each $C(i)$ is definable, and* $d\big(C \cap (C(i) \times \mathbb{K})\big) = d(C(i)) + i$, $i = 0, 1$.

Moreover, d coincides with dim *on $\mathcal{D}_\Sigma$ sets.*

See [For10a] for several cases when we know that the above conjecture holds. Notice that if a function d as in Conjecture 7.1 exists, it is unique, and satisfies:

If $X \subset \mathbb{K}$ is definable, then $d(X) = 1$ iff $F(X^4) = \mathbb{K}$, where

$$F(x_1, x_2, y_1, y_2) := \begin{cases} \frac{x_1 - x_2}{y_1 - y_2} & \text{if } y_1 \neq y_2; \\ 0 & \text{otherwise.} \end{cases}$$

Conjecture 7.1 implies the following conjecture.

Conjecture 7.2. *Let C be a $\mathcal{D}_\Sigma$ set, and $f : C \to \mathbb{K}^n$ be definable (but not necessarily continuous). Then, $\dim(f(C)) \leq \dim(C)$. In particular, there is no surjective definable function between $\mathbb{K}^n$ and $\mathbb{K}^{n+1}$.*

See Corollary 4.9 for a partial result in the direction of the above conjecture.

We introduced before i-minimal expansions of $\bar{\mathbb{R}}$: the definition extends to DC structures in the obvious way. For more on i-minimal structures outside $\mathbb{R}$, see [For10a].

Conjecture 7.3. *The open core of $\mathbb{K}$ is i-minimal.*

See lemmas 3.5 and 4.1 for partial results in the direction of the above conjecture.
The above conjectures are open even for the case when $\mathbb{K}$ expands $\bar{\mathbb{R}}$.

8. Pathologies

In this section I will give a brief exposition (far from a complete one) of "pathological" phenomena in restrained structures, which may contradict our choice of the nomenclature "restrained".

(I) The structure $\langle \bar{\mathbb{R}}, \mathbb{N} \rangle$ is wild, the class of its definable sets coincides with the class of projective sets [Kec95, Exercise 37.6]: (thus, a descriptive set theorist might say that $\langle \bar{\mathbb{R}}, \mathbb{N} \rangle$ is not *that* wild, after all).

(II) Let F be a proper real closed subfield of $\mathbb{R}$, and $\langle \bar{\mathbb{R}}, F \rangle$ be the expansion of $\bar{\mathbb{R}}$ with a unary predicate for F. Then, $\langle \bar{\mathbb{R}}, F \rangle$ has o-minimal open core, and thus it is restrained ([vdD98a,DMS10]). Notice that F is definable subset of $\mathbb{R}$ which is both dense and codense.

i) If we take F is countable, then F is an $\mathcal{F}_\sigma$ set which is not $\mathcal{D}_\Sigma$ (in $\langle \bar{\mathbb{R}}, F \rangle$).

ii) If we take F not Lebesgue measurable (resp., not projective), we have an example of a restrained structure ($\langle \bar{\mathbb{R}}, F \rangle$) which defines a set which is not Lebesgue measurable (resp., not projective).[2]

(III) [FKMS10] give an example of an i-minimal (and hence restrained) expansion of $\bar{\mathbb{R}}$ that defines a Borel isomorph of $\langle \bar{\mathbb{R}}, \mathbb{N} \rangle$.

The next examples are about restrained structures outside the reals.

(IV) There exists an ultraproduct $\mathbb{K}$ of o-minimal structures, such that $\mathbb{K}$ has the Independence Property (notice that $\mathbb{K}$ will be locally o-minimal, and hence restrained).

(V) In [HP07] Hrushovski and Peterzil produce an o-minimal structure $\mathbb{K}$ (outside $\mathbb{R}$) and a first order sentence which is true in $\mathbb{K}$ but fails in any possible interpretation over the field of real numbers.

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